M7210 Lecture 35 Monday, November 12, 2012

$\mathbb{F}[x]$-modules: Analysis of single linear map

Suppose V is a finite-dimensional vector space over $\mathbb{F}$ and $L : V \to V$ is a linear map. As explained in Lecture 33, we can view V as an $\mathbb{F}[x]$-module, where the action of x on V is defined by $xv = L(v)$. Let denote (V, L) denote the $\mathbb{F}[x]$-module defined this way. In the following, when I speak of V, I mean the $\mathbb{F}$ vector-space, and when I refer to (V, L), I mean the $\mathbb{F}[x]$ module.

In general,(V, L) is *not* cyclic. This is quite obvious if L is the zero map, for then the $\mathbb{F}[x]$-submodule of (V, L) generated by any non-zero $v_0 \in V$ is just the one-dimensional subspace spanned by v_0. But though (V, L) may not be cyclic, it is at least finitely-generated as an $\mathbb{F}[x]$-module, since any spanning set for V as an $\mathbb{F}$-vector space will obviously generate (V, L) as an $\mathbb{F}[x]$-module.

The rest of this lecture is based on N. Jacobson, *Basic Algebra I*, 2^{nd} ed., pages 195-6. Let $\mathcal{V} := \{v_0, \ldots, v_n\}$ be a basis for V. We cannot expect $\mathcal{V}$ to be independent over $\mathbb{F}[x]$. However, we have a surjection

$$\Phi : \mathbb{F}[x]^n \to (V, L)$$

of $\mathbb{F}[x]$-modules defined by $\Phi(e_i) = v_i$, where $\mathcal{E} = \{e_1, \ldots, e_n\}$ is the standard basis of the free $\mathbb{F}[x]$-module $\mathbb{F}[x]^n$. This choice of bases enables us "lift" L to $\mathbb{F}[x]^n$. If $L(v_i) = \sum_{j=1}^{n} a_{ij} v_j$ (i.e., $(L; \mathcal{V}\mathcal{V}) = (a_{ij})$), then we define $\widehat{L}(e_i) = \sum_{i=1}^{n} a_{ij} e_j$. This determines $\widehat{L}$ as an $\mathbb{F}[x]$-module endomorphism of $\mathbb{F}[x]^n$. Observe that $\Phi\widehat{L} = L\Phi$:

$$\begin{array}{cccccc} & \mathbb{F}[x]^n & \xrightarrow{\Phi} & (V, L) & \\ \widehat{L} & \downarrow & & \downarrow & L \\ & \mathbb{F}[x]^n & \xrightarrow{\Phi} & (V, L) & \end{array}$$

Let

$$\mathbf{k}_i := xe_i - \widehat{L}(e_i) \in F[x]^n,\ i = 1, \ldots, n. \qquad (*)$$

The $\mathbf{k}_i$ all belong to the kernel of Φ. Indeed, $\Phi(xe_i) = xv_i \overset{(**)}{=} L(v_i) = \Phi(\widehat{L}(e_i))$, $(**)$ being the definition of the $\mathbb{F}[x]$-module structure in (V, L). It is not obvious that the $\mathbf{k}_i$ are actually an $\mathbb{F}[x]$-module basis for $\ker \Phi$, but we now prove this.

Lemma. $\ker \Phi$ *is generated as an* $\mathbb{F}[x]$*-module by the* $\mathbf{k}_i$.

Proof. Suppose that $\mathbf{g} = \sum_{i=1}^{n} g_i(x) e_i \in \ker \Phi$. We must show that $\mathbf{g}$ is in the $\mathbb{F}[x]$-submodule of $\mathbb{F}[x]^n$ generated by the $\mathbf{k}_i$. Now, by $(*)$, $xe_i = \mathbf{k}_i + \widehat{L}(e_i)$. This implies that each $g_i(x)e_i$ can be written in the form $\sum_{i=1}^{n} h_i(x)\mathbf{k}_i + \sum_{i=1}^{n} c_i e_i$, where $c_i \in \mathbb{F}$. (This is the crux of the proof, and it is indeed a strong statement. To understand what is being claimed, write out a proof in the case that $g_1(x) = ax^2 + bx + c$, $a, b, c \in \mathbb{F}$.) Since each $g_i(x)e_i$ can be written this way, so can $\mathbf{g}$ itself. But if $\mathbf{g} = \sum_{i=1}^{n} h_i(x)\mathbf{k}_i + \sum_{i=1}^{n} c_i e_i \in \ker \Phi$, then $\sum_{i=1}^{n} c_i e_i \in \ker \Phi$, and hence $\sum_{i=1}^{n} c_i v_i = 0$. But then, each $c_i = 0$, since the v_i are independent. /////

Lemma. *The* $\mathbf{k}_i$ *are independent over* $\mathbb{F}[x]$, *and hence they form a basis for* $\ker\Phi$.

Proof. Suppose $\sum_{i=1}^n h_i(x)\mathbf{k}_i = 0$. Then

$$\sum_{i=1}^n h_i(x)xe_i = \sum_{i=1}^n h_i(x)\sum_{j=1}^n a_{ij}e_j = \sum_{j=1}^n\sum_{i=1}^n h_i(x)a_{ij}e_j,$$

and since the e_i are independent,

$$h_i(x)x = \sum_{j=1}^n h_j(x)a_{ji}.$$

But this is impossible unless all the $h_i(x)$ vanish, for if any does not, let $h_r(x)$ be the one of highest degree. Then $h_r(x)x = \sum_{j=1}^n h_j(x)a_{jr}$, but this is impossible since the left hand side has higher degree than the right. Thus, $h_i(x) = 0$ for all i. /////

The two lemmas show that $\ker\Phi$ is a free $\mathbb{F}[x]$-module of rank n. If we let

$$\mathcal{K} = \{\, \mathbf{k}_i \mid i = 1, \ldots, n \,\},$$

and let ι denote the containment $\ker\Phi \subset \mathbb{F}[x]^n$, then

$$(\iota;\mathcal{K}\mathcal{E}) = \begin{pmatrix} x-a_{11} & -a_{12} & \cdots & -a_{1,n} \\ -a_{21} & x-a_{22} & \cdots & -a_{2,n} \\ \vdots & \vdots & \cdots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & x-a_{n,n} \end{pmatrix} = xI - A.$$

In the next lecture, we will see what use we can make of this matrix.